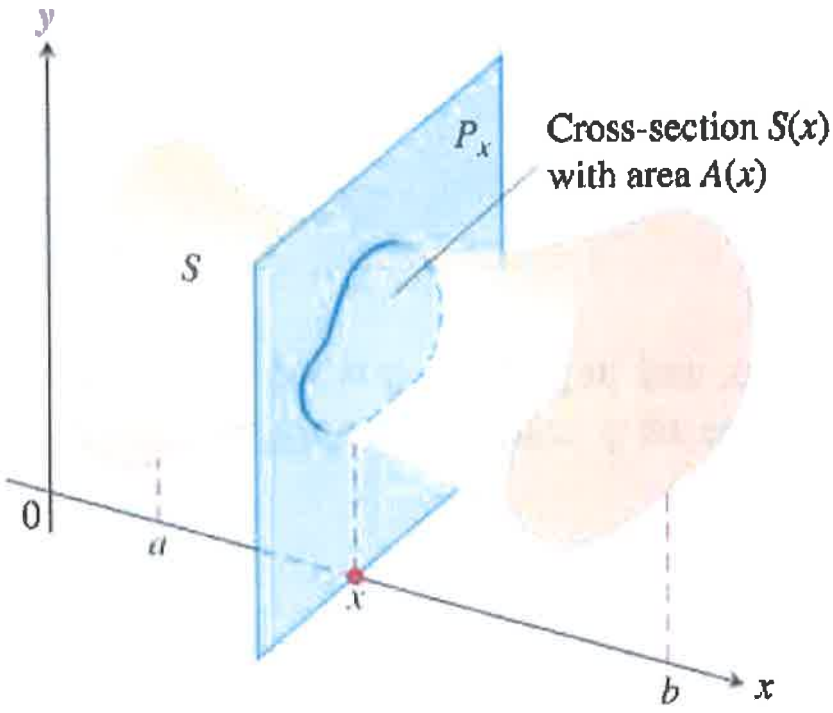
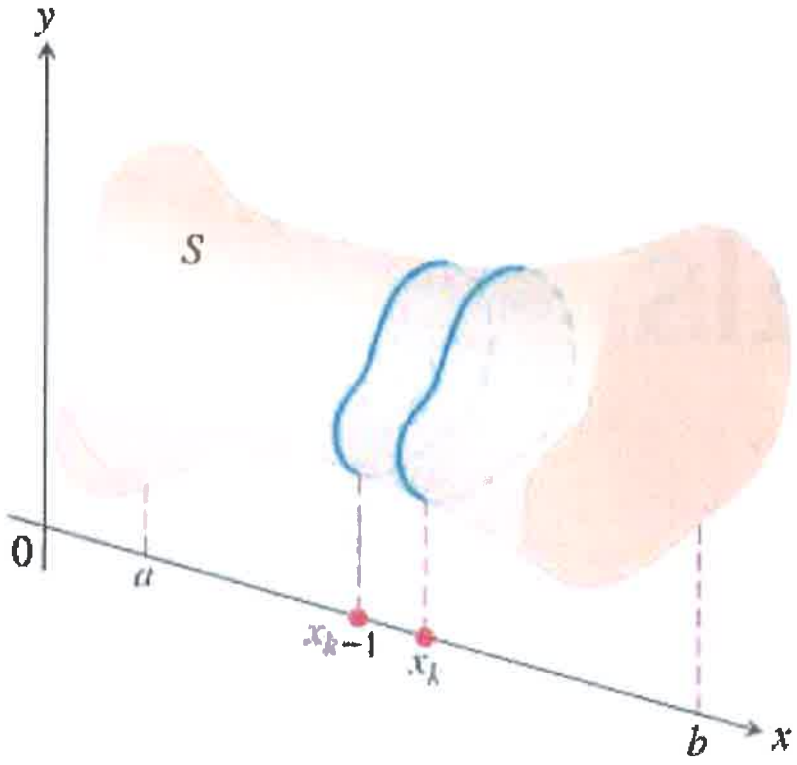
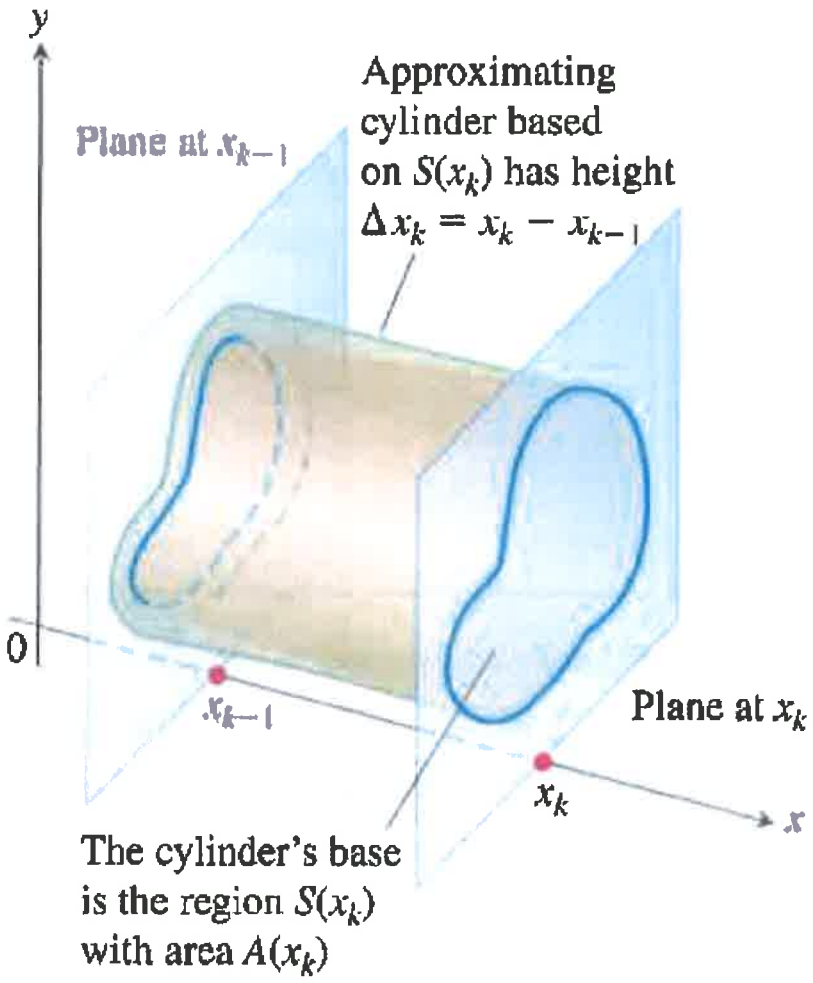
Volume of Solids: Method of Cross Sections



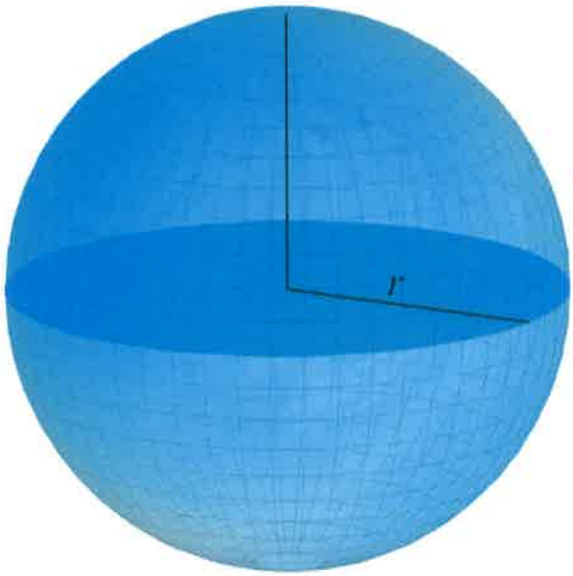




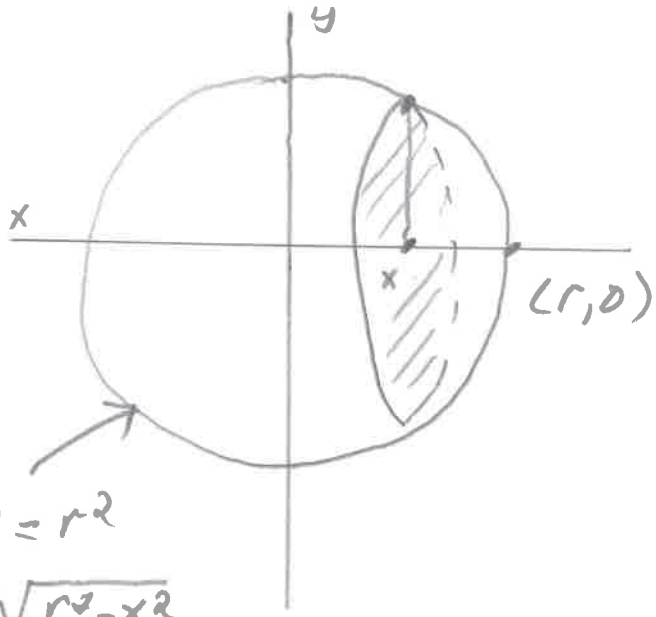
Volume of S is equal to $\lim_{\|P\|\to 0} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx$.

Example 1.

Find the volume of a sphere of radius r.



Consider cross sections.



$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

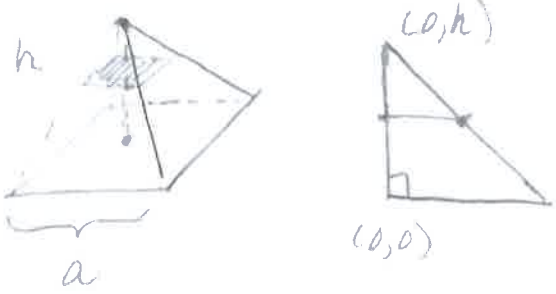
$$A(x)=\pi\left(\sqrt{r^2-x^2}\right)^2=\pi(r^2-x^2)$$

$$V=2\int_0^r \pi(r^2-x^2)\,dx=2\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r$$

$$=2\pi\left(r^3-\frac{r^3}{3}\right)=\frac{4}{3}\pi r^3.$$

Example 2.

Find the volume of a pyramid with height h and a square base with length a on each side.



$$y-h=\frac{h-0}{0-\frac{a}{2}}(x-0)$$

$$y-h=-\frac{2h}{a}x$$

$$x=\frac{a(y-h)}{-2h}$$

$$A(y)=\left[\frac{a(h-y)}{h}\right]^2,\quad 0\le y\le h$$

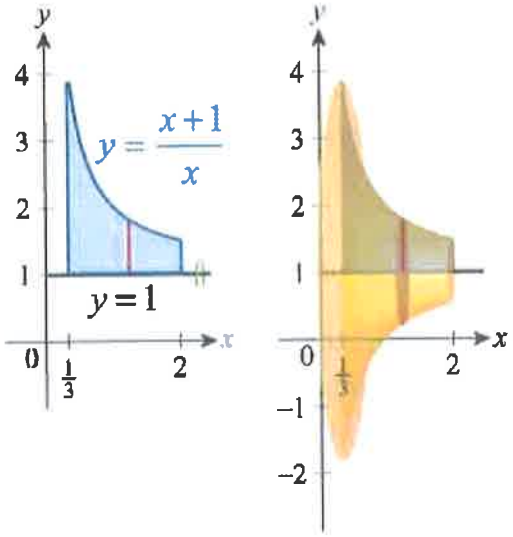
$$\text{Volume}=\int_0^h \frac{a^2}{h^2}(h-y)^2\,dy$$

$$=-\frac{a^2}{h^2}\frac{(h-y)^3}{3}\Big|_{y=0}^{y=h}=\frac{a^2}{h^2}\cdot\frac{h^3}{3}=\frac{1}{3}a^2h$$

$$V=\frac{1}{3}(\text{area of base})(\text{height})$$

Example 3.

Find the volume of the solid formed by revolving the region bounded by $y=\frac{x+1}{x}$, $y=1$, $x=\frac{1}{3}$, and $x=2$ about the line $y=1$.



$\frac{1}{3} \le x \le 2$

$A(x) = \pi\left[\frac{x+1}{x} - 1\right]^2$

$A(x) = \pi \frac{1}{x^2}$

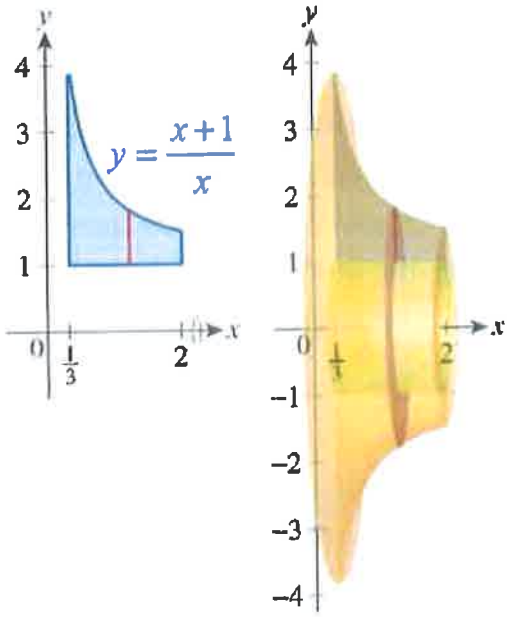
$V = \int_{\frac{1}{3}}^{2} \pi \frac{1}{x^2}\, dx$

$= \pi\left(\frac{-1}{x}\right)\Big|_{\frac{1}{3}}^{2} = \pi\left[-\frac{1}{2} + 3\right]$

$= \frac{5\pi}{2}$

Example 4.

Find the volume of the solid formed by revolving the region bounded by $y=\frac{x+1}{x}$, $y=1$, $x=\frac{1}{3}$, and $x=2$ about the x-axis.



$\frac{1}{3} \le x \le 2$

$A(x) = \left[\pi\left(\frac{x+1}{x}\right)^2 - \pi(1)^2\right]$

$A(x) = \pi\left[\frac{x^2+2x+1}{x^2} - 1\right]$

$A(x) = \pi\left[\frac{2}{x} + \frac{1}{x^2}\right]$

$V = \int_{\frac{1}{3}}^{2} \pi\left[\frac{2}{x} + \frac{1}{x^2}\right] dx$

$V = \pi\left(2\ln x - \frac{1}{x}\right)\Big|_{\frac{1}{3}}^{2}$

$V = \pi\left[\left(2\ln 2 - \frac{1}{2}\right) - \left(-2\ln 3 - 3\right)\right]$

$V = \pi\left(2\ln 2 + 2\ln 3 + \frac{5}{2}\right)$